STABILITY OF THE EQUILIBRIUM STATES OF NONHOLONOMIC SYSTEMS

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PMM Vol.29, № 1, 1965, pp.46-53

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(Received June 19, 1964)

The problem of the stability of the equilibrium states of nonholonomic systems has been considered in the works of Whittaker [1], Bottema [2], Aiserman and Gantmacher [3], Obmorshev [4], Kniazev [5] and others. However, the methods of investigating the stability as proposed in these papers, and the point of view regarding the nature of the zero roots, are not quite consistent. As is well known, Whittaker considered it possible to integrate the linearized equations of nonholonomic constraints, after which the difference between holonomic and nonholonomic systems would disappear. Essentially, Whittaker reduced the stability problem of the equilibrium states of a nonholonomic system to the corresponding problem for a holonomic system in which the number of generalized coordinates was decreased by the number of nonholonomic constraint equations. Bottema cast doubts on Whittaker's deductions and, by eliminating the inaccuracies in the reasoning, showed that in contrast to a holonomic system the characteristic determinant of a nonholonomic system is asymmetric and that the characteristic equation of a nonholonomic system has zero roots equal in number to the number of holonomic constraint equations. In view of this, Bottema concluded that here occurs a critical case of the stability theory for isolated equilibrium state. Consequently, the question of the stability of the equilibrium of a nonholonomic system in the linear approximation still remains open because to-date we do not know the general conditions for the stability of systems whose characteristic equation has an arbitrary number of zero roots. However, Aiserman and Gantmacher noted that in the given case the problem has been completely solved. They showed that this problem reduces to a special case which has been completely investigated by A.M.Liapunov and I.G.Malkin. Aiserman and Gantmacher established that the equilibrium state of nonholonomic system is stable (but not asymptotically) if all the roots of the characteristic equation, besides the zero roots which in number equal the number of nonholonomic constraint equations, have negative real parts. Relying on this result, Kniazev suggested that we should consider as critical only those cases in which the number of zero roots of the characteristic equation is larger than the number of nonholonomic constraint equations. In his own paper [5], Kniazev studied the case where the number of zero roots was one more than the number of nonholonomic constraint equations. Finally, we mention the paper [4] of Obmorshev who considered the linearized equations of small oscillations of a nonholonomic system near the equilibrium state in the general case, and also the equations of small oscillations relative to the stationary motion of a Chaplygin system. With regard to the zero roots, Obmorshev noted that Bottema had not integrated the linearized nonholonomic constraint equations and as a result had obtained unjustified roots of the characteristic equation.

The above survey of the literature indicates not only the lack of a single approach to the question of the stability of the equilibrium states of nonholonomic systems, but also the contradictions in the methods of investigating the stability. Indeed, if Whittaker was correct in integrating the linearized nonholonomic constraint equations, then Bottema was wrong in the results he obtained on the zero roots. However, if Bottema was right, then Whittaker made a fundamental error in his investigation of the stability of the equilibrium of nonholonomic systems. But then, there remains the ambiguity in the interpretation of the nature of the zero roots: Bottema, Aiserman and Gantmacher associated the appearance of the zero roots with the critical case in the sense of Liapunov, whereas Kniazev did not consider this case to be critical. Obmorshev treated the appearance of zero roots as an ambiguity which arises because the linearized nonholonomic constraint equations are not integrated.

In the present paper we shall show that a nonholonomic system has the peculiarity that its equilibrium states cannot be isolated but form a manifold whose dimension is not less than the number of nonholonomic constraints. This peculiarity stipulates the presence of zero roots in the characteristic equation. A theorem is formulated on the asymptotic stability of the manifold of equilibrium states. Examples illustrate the above statement.

1. Equilibrium state manifold of a nonholonomic system. Let the motion of a system with the Lagrange function

$$L = L(q_1, \ldots, q_n; \quad \dot{q}_1, \ldots, \dot{q}_n)$$

and the generalized forces

$$Q_1(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n), \ldots, Q_n(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

be subject to nonholonomic constraints expressec by Equations (*)

$$\omega_{\alpha\beta}(q_1, \ldots, q_n)\, \dot{q}_\beta = 0 \qquad (\alpha = 1, \ldots, m;\ \beta = 1, \ldots, n) \tag{1.1}$$

Let us set up the equations of motion with undetermined multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\beta} - \frac{\partial L}{\partial q_\beta} = Q_\beta + \lambda_\alpha \omega_{\alpha\beta} \tag{1.2}$$

The system of Equations (1.1) and (1.2) permits us to determine the $q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_m$ as functions of time and of the initial values. It follows from Equations (1.1) and (1.2) that the equilibrium states of a nonholonomic system are determined by the n equations

$$\partial L / \partial q_\beta + Q_\beta + \lambda_\alpha \omega_{\alpha\beta} = 0 \tag{1.3}$$

in the $n + m$ unknowns $q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_m$. By virtue of this, in the general case we have a manifold of equilibrium states which forms a m-dimensional surface O_m in the n-dimensional configuration space. Indeed, by expressing the generalized coordinates $q_1, \ldots, q_n$ in terms of $\lambda_1, \ldots, \lambda_m$ by means of Equations (1.3), we obtain the surface O_m in the parametric representation $q_\beta^\circ = q_\beta^\circ(\lambda_1, \ldots, \lambda_m)$, $(\beta = 1, \ldots, n)$. Let us note that in actual problems (see the examples given below) not all of Equations (1.3) may turn out to be independent. In such a case the dimension of the equilibrium state manifold will be greater than m.

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*) Here and in what follows a twice-repeated index implies summation, and a dot on a letter or on a paranthesis denotes differentiation with respect to time.

Let us eliminate the undetermined multipliers from Equations (1.2) and write the equations of motion (1.1) and (1.2) of the nonholonomic system in the normal form

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_{2n-m}) \qquad (i = 1, \ldots, 2n - m) \tag{1.4}$$

where by x_i we denote the variables $q_1, \ldots q_n, \dot{q}_1, \ldots, \dot{q}_n$. In the phase space $(x_1, \ldots, x_{2n-m})$ let the surface O_m be defined by Equations

$$x_i^\circ = x_i^\circ \; (u_1, \ldots, u_m) \quad (i = 1, \ldots, 2n - m).$$

Along with the variables $u_1, \ldots, u_m$ let us introduce the new variables $v_1, \ldots, v_{2(n-m)}$ by the relations

$$x_i = x_i^\circ (u_1, \ldots, u_m) + \gamma_{ij}(u_1, \ldots, u_m)\, v_j$$

$$(i = 1, \ldots, 2n - m;\; j = 1, \ldots, 2(n - m))$$

In the new variables Equations (1.4) are written as

$$du_i \,/\, dt = g_i(u, v), \qquad dv_j \,/\, dt = g_j(u, v) \tag{1.5}$$

Let us linearize the equations of motion (1.5) in the neighborhood of the equilibrium state surface. Expanding the right-hand sides of Equations (1.5) in a series of the small quantities $v_1, \ldots, v_{2(n-m)}$, we get

$$\begin{aligned} \frac{du_i}{dt} &= a_i(u_1, \ldots, u_m) + a_{ij}(u_1, \ldots, u_m)\, v_j + O(\|v\|^2) + \ldots \\ \frac{dv_j}{dt} &= b_j(u_1, \ldots, u_m) + b_{jk}(u_1, \ldots, u_m)\, v_k + O(\|v\|^2) + \ldots \end{aligned} \quad \begin{pmatrix} i = 1, \ldots, m \\ j,\, k = 1, \ldots, 2(n - m) \end{pmatrix}$$

$$\|v\| = (v_1^2 + v_2^2 + \ldots + v_{2(n-m)}^2)^{1/2} \tag{1.6}$$

Here the expression $O(\|v\|^2)$ denotes terms of not lower than the second order of smallness in $\|v\|$ It is not difficult to see that in Equations (1.6) the expansion coefficients a_i and b_j are equal to zero because the quantities

$$v_1, \; v_2, \ldots, v_{2(n-m)}, \; \dot{v}_1, \ldots, \dot{v}_{2(n-m)}, \; \dot{u}_1, \ldots, \dot{u}_m$$

vanish on the surface O_m. The characteristic equation of system (1.6) for any point of surface O_m has the form

$$\begin{vmatrix} p & 0 & \ldots & 0 & a_{11} & a_{12} & \ldots & a_{1,\,2(n-m)} \\ 0 & p & \ldots & 0 & a_{21} & a_{22} & \ldots & a_{2,\,2(n-m)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & p & a_{m1} & a_{m2} & \ldots & a_{m,\,2(n-m)} \\ 0 & 0 & \ldots & 0 & b_{11} - p, & b_{12} & \ldots & b_{1,\,2(n-m)} \\ 0 & 0 & \ldots & 0 & b_{21} & b_{22} - p & \ldots & b_{2,\,2(n-m)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & b_{2(n-m),1} & b_{2(n-m),2} & \ldots & b_{2(n-m),\,2(n-m)} - p \end{vmatrix} = 0 \tag{1.7}$$

from which the presence of m zero roots is immediately apparent.

Thus, the number of the zero roots of the characteristic equation is not less than the dimension of the equilibrium state manifold (*).

2. Asymptotic stability theorem for the equilibrium state manifold. From the preceding discussion it follows that it is meaningful to study the stability of the equilibrium states of a nonholonomic system only with respect to small deviations from the surface O_m

In connection with this it is natural to consider the second group of equations in system (1.6) independently of the first group of equations, temporarily treating the variables $u_1, \ldots, u_m$ as parameters.

The characteristic polynomial of this auxiliary system differs from determinant (1.7) only in the absence of the factors p^m.

Let us assume that in some region G the values of $u_1, \ldots, u_m$, the equilibrium state* $v_1 = v_2 = \ldots = v_{2(n-m)} = 0$ of the system of equations

$$dv_j / dt = b_{jk}(u_1, \ldots, u_m)\, v_k \qquad (j, k = 1, \ldots, 2(n-m)) \tag{2.1}$$

is asymptotically stable, so that

$$\|v\| < M \|v^\circ\| e^{-\sigma' t} \qquad (\sigma > 0,\ 0 < M < \infty)$$

Here the v_j° are the initial values of the variables v_j. Then, there holds the following theorem on the asymptotic stability of the equilibrium state manifold of a nonholonomic system.

T h e o r e m . Let the initial values

$$u_1^\circ, \ldots, u_m^\circ, v_1^\circ, \ldots, v^\circ_{2(n-m)}$$

be such that the values $u_1^\circ, \ldots, u_m^\circ$ lie inside the region G of asymptotic stability of Equations (2.1) and that the magnitudes of

$$v_1^\circ, \ldots, v^\circ_{2(n-m)}$$

are sufficiently small.

Then, by virtue of the equations of motion of a nonholonomic system

$$\frac{du_i}{dt} = g_i(u, v), \qquad \frac{dv_j}{dt} = g_j(u, v) \qquad (i = 1, \ldots, m;\ j = 1, \ldots, 2(n-m)) \tag{2.2}$$

the limit relations

$$\lim_{t \to +\infty} v_j(t) = 0, \qquad \lim_{t \to +\infty} u_i(t) = u_i^*$$

are satisfied, where $u_i^* \in O_m$, but in general $u_i^* \neq u_i^\circ$.

Here, for the variables $v_j(t)$ we have the estimate

$$\|v(t)\| < M' \|v^\circ\| e^{-\sigma' t} \qquad (0 < \sigma' < \sigma,\ 0 < M' < \infty) \tag{2.3}$$

P r o o f . Let us write Equations (2.2) in the form

$$\begin{aligned} du_i / dt &= \{a_{ij}(u_1, \ldots, u_m) + \Delta a_{ij}\}\, v_j \\ dv_j / dt &= \{b_{jk}(u_1, \ldots, u_m) + \Delta b_{jk}\}\, v_k \end{aligned} \qquad \begin{pmatrix} i = 1, \ldots, m \\ j, k = 1, \ldots, 2(n-m) \end{pmatrix} \tag{2.4}$$

*) The case when the number of zero roots of the characteristic equation is larger than the dimension of the equilibrium state manifold O_m should be considered as a special case.

where $|\Delta a_{ij}| < \epsilon$ and $|\Delta b_{jk}| < \epsilon$ only if for sufficiently small $\delta(\epsilon), (\delta(\epsilon) > 0)$, the inequalities

$$\| u - u^\circ \| < \delta (\varepsilon), \qquad \| v \| < \delta (\varepsilon) \tag{2.5}$$

are satisfied.

As long as inequalities (2.5) are satisfied, the estimate (*)

$$\| v (t) \| < M' \| v^\circ \| e^{-\sigma' t} \qquad (\sigma' > 0) \tag{2.6}$$

holds for the solution of Equations (2.4) for a sufficiently small value $\delta = \delta^*$.

Therefore, $$\| u^\cdot \| < N \| v^\circ \| e^{-\sigma' t}, \qquad \| u (t) \| < \frac{N}{\sigma} \| v^\circ \| \tag{2.7}$$

Let the inequality

$$\| v^\circ \| < \min \left(\frac{\delta^*}{2M'}, \frac{\delta^* \sigma'}{2N}, \frac{\delta^*}{2} \right) \tag{2.8}$$

be satisfied for a selected value $\delta = \delta^*$.

At the initial instant $t = 0$ condition (2.5) is satisfied for $\delta = \delta^*/2$; therefore, because of the uniform continuity in time t of the solution, it will be satisfied over a sertain time interval $\Delta t_0 \geqslant \tau > 0$.

Hence it follows that over this time interval estimates (2.6) and (2.7) hold. After a lapse of time Δt_0, by virtue of these estimates and of inequality (2.8), the magnitude of $v(\Delta t_0)$ satisfies inequalities (2.5) with $\delta = \delta^*/2$. But then it follows that these inequalities will be satisfied over some time interval

$$\Delta t_0 + \Delta t_1 \geqslant 2\tau.$$

By continuing this argument we establish that estimates (2.6) and (2.7) hold at any instant of time $t > 0$ since if they hold for the instant $\Delta t_0 + \ldots + \Delta t_{s-1}$, they also hold for the next time interval $\Delta t_s \geqslant \tau > 0$.

The assertion of the theorem follows from the satisfaction of estimates (2.6) and (2.7) for all t.

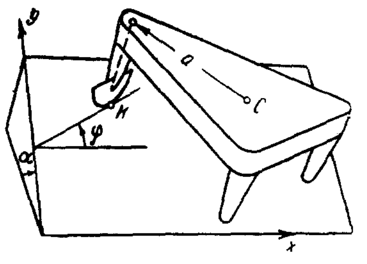

Fig. 1

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*) Indeed, according to [6], for the system of equations (2.1) there exists a positive-definite quadratic form V:

$$\alpha \| v \|^2 < V = C_{kj} v_k v_j < \beta \| v \|^2 \qquad (\alpha, \beta > 0) \tag{i}$$

such that

$$\frac{dV}{dt} < - \gamma \| v \|^2 < - \frac{\gamma}{\alpha} V \qquad (\gamma > 0) \tag{ii}$$

for arbitrary, sufficiently small variations of the coefficients b_{jk}, i.e. when the inequalities (2.5) are satisfied with sufficiently small ϵ. From (i) and (ii) we get

$$V < V |_{t=0} \, e^{-(\beta/\alpha)t}$$

and, consequently,

$$\alpha \| v \|^2 < \gamma \| v^\circ \|^2 e^{-(\beta/\alpha)t}, \quad \text{or} \quad \| v \| < \left(\frac{\gamma}{\alpha} \right)^{1/2} \| v^\circ \| e^{-(\beta/2\alpha)t}$$

which is what was required.

3. Examples. As the first example let us consider the motion of a rigid body parallel to an inclined plane. Let the body rest on the inclined plane on three legs, two of which are absolutely smooth while the third leg is equipped with a semicircular blade, consequently, the third leg cannot be displaced in the direction perpendicular to the plane of the blade.

We consider the case when the projection of the body's center of gravity onto the inclined plane lies on the straight line perpendicular to the blade and passing through the point K of contact of the blade with the inclined plane (Fig.1).

The generalized coordinates of the body are the coordinates x and y of the point K and the angle φ .

The Lagrange function is

$$L = {}^1/_2\, m\, [(x^{\cdot} + a\varphi^{\cdot} \cos\varphi)^2 + (y^{\cdot} + a\varphi^{\cdot} \sin\varphi)^2 + k^2\varphi^{\cdot 2}] - mg \sin\alpha\, (y - a\cos\varphi)$$

where m is the mass of the body and k its radius of gyration.

We introduce the dissipation function

$$\Phi = {}^1/_2\, mh\, (x^{\cdot 2} + y^{\cdot 2}) + {}^1/_2\, mh_1\varphi^{\cdot 2}$$

where $h \geqslant 0$ and $h_1 \geqslant 0$ are the viscous sliding and rotational friction coefficients, respectively. The nonholonomic constraint is expressed by Equation

$$y^{\cdot} - x^{\cdot}\tan\varphi = 0 \tag{3.1}$$

From the consideration of inertia we set up the equations of motion of the body

$$(x^{\cdot} + a\varphi^{\cdot}\cos\varphi)^{\cdot} + hx^{\cdot} + \lambda\tan\varphi = 0 \tag{3.2}$$

$$(y^{\cdot} + a\varphi^{\cdot}\sin\varphi)^{\cdot} + hy^{\cdot} + g\sin\alpha - \lambda = 0$$

$$a\,(x^{\cdot}\cos\varphi + y^{\cdot}\sin\varphi)^{\cdot} + (a^2 + k^2)\,\varphi^{\cdot\cdot} + h_1\varphi^{\cdot} + ga\sin\alpha\,\sin\varphi = 0$$

From (3.1) and (3.2) we get the equations for the equilibrium states

$$\lambda\tan\varphi = 0, \qquad \lambda = g\sin\alpha, \qquad \sin\varphi = 0 \tag{3.3}$$

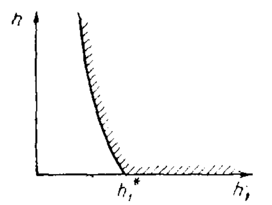

Fig. 2

Thus, the equilibrium states from the two planes: $\varphi = 0$ and $\varphi = \pi$, whose dimensions are two, even though there is only one equation of nonholonomic constraint. The increase by one in the dimension arises because two of the three equations in (3.3) sre not independent. By setting

$$x = x_0 + \xi, \quad y = y_0 + \eta$$
$$\varphi = \varphi_0 + \zeta, \quad \lambda = \lambda_0 + \theta$$

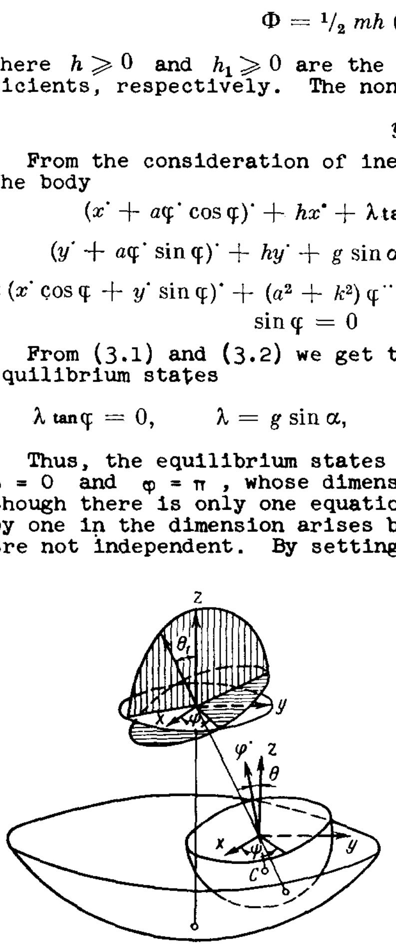

Fig. 3

where x_0, y_0, φ_0, λ_0 are the equilibrium values of the variables, we linearize the equations of motion (3.1) and (3.2)

$$\eta = 0, \quad \xi^{\cdot\cdot} + h\xi^{\cdot} \pm a\zeta^{\cdot\cdot} + g\sin\alpha\,\zeta = 0$$
$$\eta^{\cdot\cdot} + h\eta^{\cdot} - \theta = 0$$
$$\pm a\xi^{\cdot\cdot} + (a^2 + k^2)\,\zeta^{\cdot\cdot} + h_1\zeta^{\cdot} \pm ga\sin\alpha\,\zeta = 0$$

Here, the upper sign refers to the plane $\varphi = 0$ and the lower, to the plane $\varphi = \pi$. The characteristic equation is

$$p^2\,\{k^2p^3 + [h\,(a^2 + k^2) + h_1]\,p^2 + hh_1p \pm hga\sin\alpha\} = 0$$

whence it follows that the plane $\varphi = \pi$ is always unstable, while the plane $\varphi = 0$ is stable only if the inequality

$$(a^2 + k^2)\, hh_1 > k^2\, ga \sin\alpha - h^2{}_1$$

is satisfied.

The boundary of the stability region in the (h_1, h)-plane is shown in Fig. 2, where $h_1{}^* = k\sqrt{ga \sin\alpha}$.

As second example let us consider the motion of an axisymmetric body bounded below by a spherical surface of radius R, which can roll without slipping in a bowl of radius R_1. The body's center of gravity is located at a distance l from the center of its spherical surface. In the notations of Fig.3 the generalized coordinates of the body are the angles $\theta, \psi, \varphi, \theta_1, \psi_1$.

The Lagrange function is

$$\begin{aligned} L = {}^1/_2\, m\, (R_1 - R)^2\, (\theta_1^{\cdot 2} + \psi_1^{\cdot 2} \sin^2\theta_1) + {}^1/_2\, (A + ml^2)\, (\theta^{\cdot 2} + \psi^{\cdot 2}\sin^2\theta) + \\ + ml\,(R_1 - R)\, \{\psi^\cdot \psi_1^\cdot \sin\theta \sin\theta_1 \cos(\psi - \psi_1) + \theta^\cdot \theta_1^\cdot\, [\cos\theta \cos\theta_2 \cos(\psi - \psi_1) + \\ + \sin\theta \sin\theta_1] + \theta^\cdot \psi_1^\cdot \cos\theta \sin\theta_1 \sin(\psi - \psi_1) - \psi^\cdot \theta_1^\cdot \sin\theta \cos\theta_1 \sin(\psi - \psi_1)\} + \\ + {}^1/_2\, C\, (\psi^\cdot + \psi^\cdot \cos\theta)^2 + mg\, [(R_1 - R)\cos\theta - l\cos\theta_1] \end{aligned}$$

where m is the mass of the body, C its axial moment of inertia, A is the central-equatorial moment of inertia, g is the gravitational acceleration. By assuming that the dissipation of mechanical energy takes place because of the presence of friction, we introduce the dissipation function

$$\begin{aligned} \Phi = {}^1/_2\, h\, (R_1/R - 1)^2\, (\theta_1^{\cdot 2} + \psi_1^{\cdot 2} \sin^2\theta_1) + {}^1/_2\, h_1\, \{\psi^\cdot \cos\theta_1 - \theta^\cdot \sin\theta_1 \sin(\psi - \psi_1) + \\ + \varphi^\cdot\, [\sin\theta \sin\theta_1 \cos(\psi - \psi_1) + \cos\theta \cos\theta_1]\}^2 \end{aligned}$$

where $h \geqslant 0$ and $h_1 \geqslant 0$ are the viscous rolling and rotational friction coefficients, respectively. The condition of rolling without slipping leads to the two nonholonomic constraint equations

$$\begin{aligned} (R_1 - R)\, \psi_1^\cdot \sin\theta_1 + R\theta^\cdot \cos\theta_1 \sin(\psi - \psi_1) + R\psi^\cdot \sin\theta_1 + \\ + R\varphi^\cdot\, [\cos\theta \sin\theta_1 - \sin\theta \cos\theta_1 \cos(\psi - \psi_1)] = 0 \qquad (3.4) \\ (R_1 - R)\, \theta_1^\cdot + R\theta^\cdot \cos(\psi - \psi_1) + R\varphi^\cdot \sin\theta \sin(\psi - \psi_1) = 0 \end{aligned}$$

Setting up Equations (1.2) for the motion of the body by considerations of inertia, where the generalized forces will be $Q_\beta = -\partial\Phi / \partial q_\beta^\cdot$, we get the following equilibrium equations:

$$\begin{aligned} \lambda_1 \sin\theta_1 = 0, \qquad mg \sin\theta_1 = \lambda_2, \qquad \lambda_1 \sin\theta_1 = 0 \qquad (3.5) \\ mgl \sin\theta = R\lambda_1 \cos\theta_1 \sin(\psi - \psi_1) + R\lambda_2 \cos(\psi - \psi_1) \\ \lambda_1\, [\cos\theta \sin\theta_1 - \sin\theta \cos\theta_1 \cos(\psi - \psi_1)] + \lambda_2 \sin\theta \sin(\psi - \psi_1) = 0 \end{aligned}$$

From Equations (3.5) it follows that the surface O_* of the equilibrium states of the system is defined by Equations

$$\psi = \psi_1, \qquad l \sin\theta = R \sin\theta_1 \qquad (3.6)$$

and is three-dimensional. The increase by one in the dimensionality of the equilibrium state manifold arises because the first and third equations of system (3.5) are not independent. Let us introduce dimensionless quantities by means of the relations

$$\tau = t\left(\frac{g}{R}\right)^{1/2}, \quad \alpha = \frac{A}{mR^2}, \quad \beta = \frac{C}{mR^2}, \quad \gamma = \frac{l}{R}$$
$$\rho = \frac{R_1}{R}, \quad \delta = \frac{h}{mR\sqrt{gR}}, \quad \delta_1 = \frac{h_1}{mR\sqrt{gR}}$$

and let us linearize Equations (1.2) and (3.4) in a neighborhood of the equilibrium state surface (3.6). Denoting the derivation of a variable from its equilibrium value by attaching a minus sign as a subscript to the symbol of that variable, after eliminating the undetermined multipliers we obtain the following linearized equations:

$$[\alpha + \gamma^2 - \gamma \cos(\theta - \theta_1)]\, \theta_-^{\cdot\cdot} + \gamma\theta_- \cos\theta - (\rho - 1)\, [1 - \gamma \cos(\theta - \theta_1)] \times \theta_{1-}^{\cdot\cdot} - \delta(\rho - 1)\, \theta_1^{\cdot} - \cos\theta_1\theta_{1-} = 0$$

$$(\alpha \sin^2\theta + \beta \cos^2\theta)\, \psi_-^{\cdot\cdot} + \delta_1 \cos^2\theta_1\psi_-^{\cdot} - \delta(\rho - 1) \sin^2\theta_1\psi_{1-}^{\cdot} + \beta\cos\theta\varphi_-^{\cdot\cdot} + \delta_1 \cos\theta_1 \cos(\theta - \theta_1)\, \varphi_-^{\cdot} = 0$$

$$\beta\varphi_-^{\cdot\cdot} + \delta_1 \cos_2(\theta - \theta_1)\, \varphi_-^{\cdot} + (\rho - 1) \sin\theta_1 \sin(\theta - \theta_1)\, \psi_1^{\cdot\cdot} + \delta \sin\theta_1 \sin(\theta - \theta_1)\, \psi_{1-}^{\cdot} + \gamma \sin^2\theta\psi_1 + [\beta\cos\theta + \gamma \sin\theta \sin(\theta - \theta_1)]\, \psi_-^{\cdot\cdot} + \delta_1 \cos\theta_1 \cos(\theta - \theta_1)\, \psi_-^{\cdot} - \gamma \sin^2\theta\psi_- = 0$$

$$(\rho - 1) \sin\theta_1\psi_{1-} + \sin\theta_1\psi_- - \sin(\theta - \theta_1)\, \varphi_-^{\cdot} = 0$$

$$(\rho - 1)\, \theta_1^{\cdot} + \theta_-^{\cdot} = 0$$

Here, in correspondence with (3.6) we have the relation

$$\gamma \sin\theta = \sin\theta_1$$

satisfied at the equilibrium state. The characteristic equation of the system being considered takes the form

$$p^3\, [a_0p^2 + \delta(\rho - 1)\, p + a_1]\, (b_0p^3 + b_1p^2 + b_2p + b_3) = 0 \qquad (3.7)$$

Here

$$a_0 = (\rho - 1)\, [\alpha + 1 - 2\gamma \cos(\theta - \theta_1) + \gamma^2]$$

$$a_1 = \cos\theta_1 + \gamma(\rho - 1) \cos\theta$$

$$b_0 = (\rho - 1)\, (\alpha\beta + a^2b), \qquad u = \cos\theta_1 - \gamma\cos\theta, \qquad b = \alpha\sin^2\theta + \beta\cos^2\theta$$

$$b_1 = (\rho - 1)\, \{\delta\, [\beta + (\alpha - \beta + \gamma^2) \sin^2(\theta - \theta_1)] + \delta_1\, [\alpha + (\beta - \alpha) \sin^2(\theta - \theta_1) + a^2 \cos^2\theta_1]\}$$

$$b_2 = \delta\delta_1\, (\rho - 1) + \rho\beta\gamma \cos\theta + ab$$

$$b_3 = \delta_1\, [\rho\gamma \cos\theta_1 \cos(\theta - \theta_1) + a \cos^2\theta_1] - \delta\, (\rho - 1)\, a \sin^2\theta_1$$

For the parameter values $\rho > 1$, $0 < \gamma < 1$, the stability region for the system's equilibrium state manifold is determined by the inequalities

$$b_3 > 0, \qquad b_1b_2 - b_0b_3 > 0 \qquad (3.8)$$

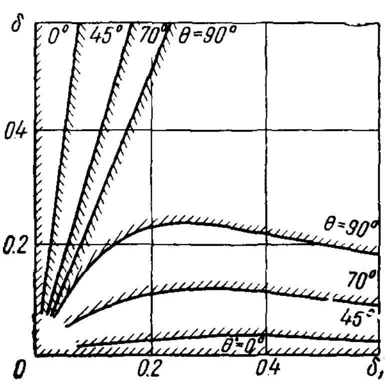

Fig. 4

Fig.4 shows the stability region boundaries in the (δ_1, δ)-plane, constructed for a homogeneous semisphere (α = 0.26, β = 0.4, γ = 0.375) which rolls without slipping in a spherical bowl. The calculations were carried out for the case when the radius of the semisphere was one-fourth the radius of the bowl.

N o t e . From Expression (3.7) it follows that as $\rho \to \infty$, where the spherical bowl degenerates into a plane, the stability conditions (3.8) are satisfied for any value of $\gamma > 0$. In particular, all the equilibrium states of a semisphere on a plabe are stable.

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Translated by N.H.C.